

THE MANIN-MUMFORD CONJECTURE FOR SUBVARIETIES WITH AMPLE COTANGENT BUNDLE

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ABSTRACT. We give a new proof of the Manin-Mumford conjecture for subvarieties of abelian varieties having ample cotangent bundle, when all data are defined over a number field. Our strategy follows Buium’s approach in the case of curves; i.e. we prove an intermediate “non-ramified version” from which the conjecture easily follows. In order to do so, we use the Greenberg transform, which assumes the role of the p -jet spaces in Buium’s work, and the theory of strongly semistable sheaves, which allows us to generalize to higher dimensions a result on the sparsity of p -divisible unramified liftings obtained by Raynaud in the case of curves. Furthermore, we provide an explicit bound for the cardinality of the set of prime-to- p torsion points of subvarieties obtained as the intersection of at least $n/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension n .

1. INTRODUCTION

The Manin-Mumford conjecture is a significant result concerning the intersection of a subvariety X of an abelian variety A with the group of torsion points of A , when all data are defined over a number field. Raynaud first proved the conjecture in 1983, and since then various other proofs (sometimes only for the case of curves) surfaced, due to Serre, Coleman, Hindry, Buium, Hrushovski, Pink-Rössler. In this paper we give a new proof of the Manin-Mumford conjecture in the case of a subvariety X with ample cotangent bundle (cf. Theorem 14):

Theorem 1. *Let K be a number field, A be an abelian variety over K and $X \subseteq A$ be a smooth subvariety over K with trivial translation stabilizer and ample cotangent bundle. Then the set*

$$\mathrm{Tor}(A(\overline{K})) \cap X(\overline{K})$$

is finite.

Here \overline{K} is a fixed algebraic closure of K and $\mathrm{Tor}(A(\overline{K}))$ is the subgroup of torsion points of $A(\overline{K})$. See the begin of the next section for the definition of translation stabilizer.

Our strategy for proving Theorem 1 follows Buium's approach in the case of curves (cf. [Bui96]). We fix a prime \mathfrak{p} of K above a prime p such that K/\mathbb{Q} is unramified at \mathfrak{p} and X has good reduction at \mathfrak{p} . The key point then is to prove a “non-ramified version” of Theorem 1: more precisely, we prove that Theorem 1 holds with \overline{K} replaced by the maximal extension of K contained in \overline{K} which is unramified above \mathfrak{p} and with the torsion replaced by the prime-to- p torsion (see Theorem 13 and the first part of the proof of Theorem 14).

This “non-ramified version” is a consequence of a result on the sparsity of p -divisible unramified liftings which holds for general subvarieties, not necessarily having ample cotangent bundle. Let U be an open subscheme of $\mathrm{Spec} \mathcal{O}_K$ not containing any ramified prime and such that A/K extends to an abelian scheme \mathcal{A}/U and X extends to a smooth closed integral subscheme \mathcal{X} of \mathcal{A} . For any $\mathfrak{p} \in U$, let R (resp. R_1) be the ring of Witt vectors (resp. of length 2) with coordinates in the algebraic closure $\overline{k(\mathfrak{p})}$ of the residue field of \mathfrak{p} . We denote by $X_{\mathfrak{p}^1}$ (resp. $A_{\mathfrak{p}^1}$) the R_1 -scheme $\mathcal{X} \times_U \mathrm{Spec} R_1$ (resp. $\mathcal{A} \times_U \mathrm{Spec} R_1$) and we denote by $X_{\mathfrak{p}^0}$ the $\overline{k(\mathfrak{p})}$ -scheme $\mathcal{X} \times_U \mathrm{Spec} \overline{k(\mathfrak{p})}$. Define \tilde{U} as the nonempty open subscheme of U consisting of all $\mathfrak{p} \in U$ such that $X_{\mathfrak{p}^0}$ has trivial translation stabilizer.

Our result on the sparsity of p -divisible unramified liftings is the following (cf. Theorem 12 for a more precise formulation):

Theorem 2. *Let n be the dimension of X . Let $\mathfrak{p} \in \tilde{U}$ be above a prime p with $p > n^2 \deg(\Omega_X)$. Then the set*

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

(Here $\deg(\Omega_X)$ refers to the degree of Ω_X computed with respect to any fixed very ample line bundle on X . See Section 4 for its definition.)

Theorem 2 is an effective form (in the case of a number field) of a result on the sparsity of highly p -divisible unramified liftings given by D. Rössler (see Theorem 4.1 in [Rös13] and the comment right after Theorem 12 in this paper). Theorem 2 is also a generalization to higher dimensions of the analogue result obtained by Raynaud in the case of curves (cf. Théorème 4.4.1 in [Ray83b]).

The proof of Theorem 2 lies on the impossibility of lifting the Frobenius of $X_{\mathfrak{p}^0}$ over R_1 . This is a well-known fact in the case of smooth curves of genus at least 2: Raynaud proved it (see Lemma I.5.4 in [Ray83a]) by means of the Cartier isomorphism. To prove the impossibility of lifting the Frobenius in higher dimensions (for

subvarieties with trivial stabilizer) we make use of the theory of strongly semistable sheaves and of the Cartier isomorphism.

Subvarieties of abelian varieties with ample cotangent bundle have been studied by O. Debarre in his article [Deb05]. In it, he proved for example that the intersection of at least $n/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension n has ample cotangent bundle. This provides us with an entire class of subvarieties of abelian varieties for which our proof of the Manin-Mumford conjecture works. For this class of subvarieties, we give an explicit bound for the set of prime-to- p torsion points:

Theorem 3. *Let K be a number field, A/K be an abelian variety of dimension n and $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c, e \in \mathbb{N}$ with $c \geq n/2$. Let $H_1, H_2, \dots, H_c \in |L^e|$ be general and let e be sufficiently big, so that the subvariety*

$$X := H_1 \cap H_2 \cap \dots \cap H_c$$

has ample cotangent bundle. Suppose X is smooth and has trivial translation stabilizer. Then there exists a nonempty open subset $V \subseteq \text{Spec}(\mathcal{O}_K)$ such that, if \mathfrak{p} is in V and \mathfrak{p} is above a prime $p > (n - c)^2 c e^{c+1} (L^n)$, the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ is bounded above by

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot Q_{n,c,h} \right) (L^n)^2$$

where $\text{Tor}^p(A(\overline{K}))$ is the set of prime-to- p torsion points of $A(\overline{K})$ and $Q_{n,c,h}$ is a natural number depending on n, c and h . (Cf. Theorem 15 for details about $Q_{n,c,h}$ and V .)

To obtain such a “quantitative version” of the Manin-Mumford conjecture (for the prime-to- p torsion) we use the same technique present in Buium’s paper. If $\text{Gr}_1(X_{\mathfrak{p}^1})$ and $\text{Gr}_1(A_{\mathfrak{p}^1})$ denote the Greenberg transform of level 1 of X and A (see Section 3), then we embed the two varieties $\text{Gr}_1(X_{\mathfrak{p}^1})$ and $[p]_* \text{Gr}_1(A_{\mathfrak{p}^1})$ into the same projective space and use Bezout’s Theorem to compute the cardinality of their intersection.

We conclude this introduction by giving an outline of the paper:

- in Section 2 we fix those notations which will stay unchanged throughout the paper;

- in Section 3 we recall definitions and basic properties of the Greenberg transform and the critical schemes. Notice that the use of the Greenberg transform in this paper corresponds to the use of p-jet spaces in Buium's proof;
- Section 4 is dedicated to the theory of strongly semistable sheaves;
- Section 5 contains the proof of Theorem 2;
- in Section 6 we prove Theorem 1;
- in Section 7 we prove Theorem 3;
- Section 8 is an Appendix containing the proof of the following fact: the tangent bundle of a smooth subvariety of an abelian variety which has trivial stabilizer has no non-zero global sections (cf. Fact 17).

2. NOTATIONS

We fix the following notations:

- K a number field,
- \overline{K} an algebraic closure of K ,
- A/K an abelian variety,
- $X \subseteq A$ a closed integral subscheme, smooth over K ,
- $\text{Stab}_A(X)$ the translation stabilizer of X in A , i.e. the closed subgroup scheme of A characterized uniquely by the fact that for any K -scheme S and any morphism $b : S \rightarrow A$, translation by b on the product $A \times_K S$ maps the subscheme $X \times_K S$ to itself if and only if b factors through $\text{Stab}_A(X)$,
- U an open subscheme of $\text{Spec } \mathcal{O}_K$ not containing any ramified prime and such that A/K extends to an abelian scheme \mathcal{A}/U and X extends to a smooth closed integral subscheme \mathcal{X} of \mathcal{A} .

For any prime number p , any $\mathfrak{p} \in U$ above p and any $n \geq 0$, we denote by:

- $k(\mathfrak{p})$ the residue field $\mathcal{O}_K/\mathfrak{p}$ for \mathfrak{p} ,
- $K_{\mathfrak{p}}$ the completion of K with respect to \mathfrak{p} ,
- $\widehat{K_{\mathfrak{p}}}^{\text{unr}}$ the completion of the maximal unramified extension of $K_{\mathfrak{p}}$,
- $R := W(\overline{k(\mathfrak{p})})$ (resp. $R_n := W_n(\overline{k(\mathfrak{p})})$) the ring of Witt vectors (resp. the ring of Witt vectors of length $n+1$) with coordinates in $\overline{k(\mathfrak{p})}$. We recall that R can be identified with the ring of integers of $\widehat{K_{\mathfrak{p}}}^{\text{unr}}$ and R_0 with $\overline{k(\mathfrak{p})}$,
- $X_{\mathfrak{p}^n}$ the R_n -scheme $\mathcal{X} \times_U \text{Spec } R_n$
 $A_{\mathfrak{p}^n}$ the R_n -scheme $\mathcal{A} \times_U \text{Spec } R_n$,
- $\text{Tor}(A(\overline{K}))$ the torsion points in $A(\overline{K})$,
- $\text{Tor}^p(A(\overline{K})) \subseteq \text{Tor}(A(\overline{K}))$ the prime-to- p torsion,

- $\text{Tor}_p(A(\overline{K})) \subseteq \text{Tor}(A(\overline{K}))$ the p -torsion.

3. THE GREENBERG TRANSFORM AND THE CRITICAL SCHEMES

Now we recall some basic facts about the Greenberg transform (for more details, see [Gre61], [Gre63] and [[BLR90] p. 276-277]).

Throughout this section, a prime number p and a $\mathfrak{p} \in U$ above p are fixed.

For any $n \geq 0$, the Greenberg transform of level n is a covariant functor Gr_n from the category of R_n -schemes locally of finite type, to the category of $\overline{k(\mathfrak{p})}$ -schemes locally of finite type. If Y_n is an R_n -scheme locally of finite type, $\text{Gr}_n(Y_n)$ is a $\overline{k(\mathfrak{p})}$ -scheme with the property

$$Y_n(R_n) = \text{Gr}_n(Y_n)(\overline{k(\mathfrak{p})}).$$

More precisely, we can interpret R_n as the set of $\overline{k(\mathfrak{p})}$ -valued points of a ring scheme \mathcal{R}_n over $\overline{k(\mathfrak{p})}$. For any $\overline{k(\mathfrak{p})}$ -scheme T , we define $\mathbb{W}_n(T)$ as the ringed space over R_n consisting of T as a topological space and of $\text{Hom}_{\overline{k(\mathfrak{p})}}(T, \mathcal{R}_n)$ as a structure sheaf. By definition $\text{Gr}_n(Y_n)$ represents the functor from the category of schemes over $\overline{k(\mathfrak{p})}$ to the category of sets given by

$$T \mapsto \text{Hom}_{R_n}(\mathbb{W}_n(T), Y_n)$$

where Hom stands for homomorphisms of ringed spaces. In other words, the functor Gr_n is right adjoint to the functor \mathbb{W}_n .

The functor Gr_n respects closed immersions, open immersions, fibre products, smooth and étale morphisms. Furthermore it sends group schemes over R_n to group schemes over $\overline{k(\mathfrak{p})}$. The canonical morphism $R_{n+1} \rightarrow R_n$ gives rise to a functorial transition morphism $\text{Gr}_{n+1} \rightarrow \text{Gr}_n$.

Let Y_n be a scheme over R_n locally of finite type. Then for any $m < n$ we can define

$$Y_m := Y_n \times_{R_n} R_m.$$

Let us call $F_{Y_0} : Y_0 \rightarrow Y_0$ the absolute Frobenius endomorphism of Y_0 and $\Omega_{Y_0/\overline{k(\mathfrak{p})}}$ the sheaf of relative differentials.

For any finite rank locally free sheaf \mathcal{F} over Y_0 we will write

$$V(\mathcal{F}) := \underline{\text{Spec}}(\text{Sym}(\mathcal{F}^\vee))$$

for the vector bundle over $\overline{k(\mathfrak{p})}$ associated to \mathcal{F} .

If Y_n is smooth over R_n , then $\Omega_{Y_0/\overline{k(\mathfrak{p})}}$ is a locally free sheaf. A key result in [Gre63] is the following structure theorem:

$$\mathrm{Gr}_1(Y_1) \rightarrow \mathrm{Gr}_0(Y_0)$$

is a torsor under

$$V \left(F_{Y_0}^* \Omega_{Y_0/\overline{k(\mathfrak{p})}} \right).$$

Let X , A , \mathcal{X} and \mathcal{A} be as fixed in the previous section. For any $n \geq 0$ we define the n -critical scheme

$$\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) := [p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n}) \cap \mathrm{Gr}_n(X_{\mathfrak{p}^n}),$$

where $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ refers to the scheme-theoretic image of $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$ by the multiplication map $[p^n]$. Notice that $\mathrm{Crit}^n(\mathcal{X}, \mathcal{A})$ is a scheme over $\overline{k(\mathfrak{p})}$ and that $\mathrm{Crit}^0(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$, since Gr_0 is the identity.

The natural morphisms $\mathrm{Gr}_{n+1}(A_{\mathfrak{p}^{n+1}}) \rightarrow \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ lead to a projective system of $\overline{k(\mathfrak{p})}$ -schemes:

$$\cdots \rightarrow \mathrm{Crit}^2(\mathcal{X}, \mathcal{A}) \rightarrow \mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) \rightarrow X_{\mathfrak{p}^0}$$

whose connecting morphisms are both affine and proper, hence finite. In fact, transition morphisms are always affine and for any $n \geq 0$ the subscheme $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ is proper, being the greatest abelian subvariety of $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$.

We shall write $\mathrm{Exc}^n(\mathcal{X}, \mathcal{A})$ for the scheme theoretic image of the morphism $\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) \rightarrow X_{\mathfrak{p}^0}$.

4. THE GEOMETRY OF VECTOR BUNDLES IN POSITIVE CHARACTERISTIC

Let us recall some results on vector bundles in positive characteristic we will need later (see paragraph 2 in [Rös14] for all the details and proofs).

Let Y be a smooth projective variety over an algebraically closed field l_0 of positive characteristic. We write as before Ω_{Y/l_0} for the sheaf of differentials of Y over l_0 and $F_Y : Y \rightarrow Y$ for the absolute Frobenius endomorphism of Y . We start with following elementary lemma.

Lemma 4. *Let*

$$0 \rightarrow V \rightarrow W \rightarrow N \rightarrow 0$$

be an exact sequence of vector bundles on Y . Suppose that $W \simeq \mathcal{O}_Y^l$ for some $l > 0$. Then for any dominant proper morphism $\phi : Y' \rightarrow Y$, where Y' is integral, the morphism

$$\phi^* : H^0(Y, V) \rightarrow H^0(Y', \phi^* V)$$

is an isomorphism.

Proof. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(Y, V) & \longrightarrow & H^0(Y, W) & \longrightarrow & H^0(Y, N) \\ & & \downarrow \phi^* & & \downarrow \phi^* & & \downarrow \phi^* \\ 0 & \longrightarrow & H^0(Y', \phi^* V) & \longrightarrow & H^0(Y', \phi^* W) & \longrightarrow & H^0(Y', \phi^* N) \end{array}$$

In this diagram all three vertical arrows are injective thanks to the surjectivity of ϕ . Furthermore, the middle vertical arrow is an isomorphism, since both $H^0(Y, \mathcal{O}_Y)$ and $H^0(Y', \mathcal{O}_{Y'})$ coincide with l_0 and both W and $\phi^* W$ are trivial. The five lemma now implies that the left vertical arrow is surjective. \square

Let L be a very ample line bundle on Y . If V is a torsion free coherent sheaf on Y , we shall write

$$\mu(V) = \mu_L(V) = \deg_L(V)/\mathrm{rk}(V)$$

for the slope of V (with respect to L). Here $\mathrm{rk}(V)$ is the rank of V , i.e. the dimension of the stalk of V at the generic point of Y . Furthermore,

$$\deg_L(V) := \int_Y c_1(V) \cdot c_1(L)^{\dim(Y)-1}$$

where $c_1(\cdot)$ refers to the first Chern class with values in an arbitrary Weil cohomology theory and the integral \int_Y stands for the push-forward morphism to $\mathrm{Spec} l_0$ in that theory. Recall that V is called semistable (with respect to L) if for every coherent subsheaf W of V , we have $\mu(W) \leq \mu(V)$ and it is called strongly semistable if $F_Y^{n,*} V$ is semistable for all $n \geq 0$.

In general, there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

of V by subsheaves, such that the quotients V_i/V_{i-1} are all semistable and such that the slopes $\mu(V_i/V_{i-1})$ are strictly decreasing for $i \geq 1$. This filtration is unique and is called the Harder-Narasimhan (HN) filtration of V . We will say that V has a

strongly semistable HN filtration if all the quotients V_i/V_{i-1} are strongly semistable. We shall write

$$\mu_{\min}(V) := \inf\{\mu(V_i/V_{i-1})\}_{i \geq 1}$$

and

$$\mu_{\max}(V) := \sup\{\mu(V_i/V_{i-1})\}_{i \geq 1}.$$

An important consequence of the definitions is the following fact: if V and W are two torsion free sheaves on Y and $\mu_{\min}(V) > \mu_{\max}(W)$, then $\text{Hom}_Y(V, W) = 0$. We also recall that we have the following equivalences:

$$V \text{ is semistable} \Leftrightarrow \mu_{\min}(V) = \mu_{\max}(V) \Leftrightarrow \mu_{\min}(V) = \mu(V).$$

The first one is clear, the second one is a consequence of:

$$\mu_{\min}(V) = \min\{\mu(Q) \mid Q \text{ is a quotient of } V\}.$$

For more on the theory of semistable sheaves, see the monograph [HL10].

The following two theorems are key results from A. Langer (cf. Theorem 2.7 and Corollary 6.2 in [Lan04]).

Theorem 5. *If V is a torsion free coherent sheaf on Y , then there exists $n_0 \geq 0$ such that $F_Y^{n,*}V$ has a strongly semistable HN filtration for all $n \geq n_0$.*

If V is a torsion free coherent sheaf on Y , we now define

$$\bar{\mu}_{\min}(V) := \lim_{r \rightarrow \infty} \mu_{\min}(F_Y^{r,*}V)/\text{char}(l_0)^r$$

and

$$\bar{\mu}_{\max}(V) := \lim_{r \rightarrow \infty} \mu_{\max}(F_Y^{r,*}V)/\text{char}(l_0)^r.$$

Note that Theorem 5 implies that the two sequences $\mu_{\min}(F_Y^{r,*}V)/\text{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)/\text{char}(l_0)^r$ become constant when r is sufficiently large, so the above definitions of $\bar{\mu}_{\min}$ and $\bar{\mu}_{\max}$ make sense. Furthermore the sequences $\mu_{\min}(F_Y^{r,*}V)/\text{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)/\text{char}(l_0)^r$ are respectively weakly decreasing and weakly increasing, therefore we have

$$\mu_{\min}(V) \geq \bar{\mu}_{\min}(V) \quad \text{and} \quad \bar{\mu}_{\max}(V) \geq \mu_{\max}(V).$$

Let us define

$$\alpha(V) := \max\{\mu_{\min}(V) - \bar{\mu}_{\min}(V), \bar{\mu}_{\max}(V) - \mu_{\max}(V)\}.$$

Theorem 6. *If V is of rank r , then*

$$\alpha(V) \leq \frac{r-1}{\text{char}(l_0)} \max \{ \bar{\mu}_{\max}(\Omega_{Y/l_0}), 0 \}.$$

In particular, if $\bar{\mu}_{\max}(\Omega_{Y/l_0}) \geq 0$ and $\text{char}(l_0) \geq d = \dim Y$,

$$\bar{\mu}_{\max}(\Omega_{Y/l_0}) \leq \frac{\text{char}(l_0)}{\text{char}(l_0) + 1 - d} \mu_{\max}(\Omega_{Y/l_0}).$$

The following lemmas will be a key input in the proof of Theorem 12.

Lemma 7. *Let V be a vector bundle over Y . Suppose that*

- *for any surjective finite morphism $\phi : Y' \rightarrow Y$, we have $H^0(Y', \phi^*V) = 0$,*
- *V^\vee is globally generated.*

Then

- *for any surjective finite morphism $\phi : Y' \rightarrow Y$, such that Y' is smooth over l_0 , we have $\mu_{\min}(\phi^*V^\vee) > 0$. In particular $\bar{\mu}_{\min}(V^\vee) > 0$;*
- *there is an $n_0 \in \mathbb{N}$ such that $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for all $n > n_0$.*

Proof. The bundle V^\vee is globally generated, so for any ϕ as in the hypotheses $\phi^*(V^\vee)$ is globally generated. This implies

$$\mu_{\min}(\phi^*(V^\vee)) \geq 0.$$

Actually $\mu_{\min}(\phi^*(V^\vee))$ cannot be zero. In fact, suppose by contradiction that $\phi^*(V^\vee)$ has a non-zero semistable quotient $Q = \phi^*(V^\vee)/Q_0$ of degree zero. Lemma 2.2 in [Rös14] shows that any globally generated torsion free sheaf of degree zero is trivial, so we have

$$\phi^*(V^\vee) \simeq Q_0 \oplus \mathcal{O}_{Y'}^{\oplus d}$$

for some $d > 0$. This implies that $\phi^*(V^\vee)$ has a non-vanishing section, which contradicts the assumptions.

To prove the second assertion it is enough to show that

$$\mu_{\min}(F_Y^{n,*}(V^\vee)) > \mu_{\max}(\Omega_{Y/l_0})$$

for n large enough, since

$$H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = \text{Hom}_Y(F_Y^{n,*}(V^\vee), \Omega_{Y/l_0}).$$

Now taking $\phi = F_Y^{n,*}$, we obtain that $\mu_{\min}(F_Y^{n,*}(V^\vee)) > 0$ for any n . Since the sequence

$$\mu_{\min}(F_Y^{n,*}(V^\vee)) / \text{char}(l_0)^n$$

becomes constant for n sufficiently large, it follows that $\mu_{\min}(F_Y^{n,*}(V^\vee))$ tends to infinity and therefore

$$\mu_{\min}(F_Y^{n,*}(V^\vee)) > \mu_{\max}(\Omega_{Y/l_0})$$

for n big enough. \square

Lemma 8. *Let V and Y be as in Lemma 7. Let n_0 verify $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for all $n > n_0$ and let $T \rightarrow Y$ be a torsor under*

$$V(F_Y^{n_0,*}V) := \underline{\text{Spec}}(\text{Sym}(F_Y^{n_0,*}V^\vee)).$$

Let $\phi : Y' \rightarrow Y$ be a proper surjective morphism and suppose that Y' is irreducible. Then we have the implication:

$$\phi^*T \text{ is a trivial } V(\phi^*(F_Y^{n_0,*}V))\text{-torsor} \implies T \text{ is a trivial } V(F_Y^{n_0,*}V)\text{-torsor}.$$

The proof of this lemma uses Lemma 7, an argument attributed to Moret-Bailly in order to restrict to the case in which ϕ is generically purely inseparable and a result by Spziri, Lewin-Ménégaux stating the injectivity of the map

$$H^1(Y, V) \rightarrow H^1(Y, F_Y^*V)$$

whenever $H^0(Y, F_Y^*V \otimes \Omega_{Y/l_0}) = 0$. See Corollary 2.8 in [Rös14] for the actual proof.

5. SPARSITY OF \mathfrak{p} -DIVISIBLE UNRAMIFIED LIFTINGS

Let K , A and X be as fixed in Notations and let $\text{Stab}_A(X)$ be trivial.

In this section we prove our result on the sparsity of \mathfrak{p} -divisible unramified liftings (cf. Theorem 12 below). A fundamental intermediate step to do so will be Lemma 10.

The construction of the stabilizer commutes with the base change, so we have

$$\text{Stab}_A(X) = \text{Stab}_A(\mathcal{X}) \times_U \text{Spec} K.$$

Since $\text{Stab}_A(X)$ is trivial, by generic flatness and finiteness, we can restrict the map $\pi : \text{Stab}_A(\mathcal{X}) \rightarrow U$ to the inverse image of a non-empty open subscheme $U' \subset U$ to obtain a finite flat commutative group scheme of degree one

$$\pi|_{\pi^{-1}(U')} : \pi^{-1}(U') \rightarrow U'.$$

Corollary 3 in paragraph 4 of [SCA86] implies that $\pi|_{\pi^{-1}(U')}$ is étale and therefore an isomorphism. In particular, for any $\mathfrak{q} \in U'$ we have that $\text{Stab}_{A_{\mathfrak{q}_0}}(X_{\mathfrak{q}_0})$ is

trivial. Therefore there exist only finitely many $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ elements in U such that $\text{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0})$ is not trivial. We will denote by $\tilde{U} \subseteq U$ the open subscheme

$$\tilde{U} := U \setminus \{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}.$$

For any $\mathfrak{p} \in U$ we denote by $F_{\overline{k(\mathfrak{p})}}$ the Frobenius automorphism on $\overline{k(\mathfrak{p})}$ and by F_{R_1} the automorphism of R_1 induced by $F_{\overline{k(\mathfrak{p})}}$. We define

$$\begin{aligned} X'_{\mathfrak{p}^0} &:= X_{\mathfrak{p}^0} \times_{F_{\overline{k(\mathfrak{p})}}} \overline{k(\mathfrak{p})} \\ X'_{\mathfrak{p}^1} &:= X_{\mathfrak{p}^1} \times_{F_{R_1}} R_1 \end{aligned}$$

and we write

$$F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}} : X_{\mathfrak{p}^0} \rightarrow X'_{\mathfrak{p}^0}$$

for the relative Frobenius on $X_{\mathfrak{p}^0}$. For brevity's sake, from now on we will write

$$\Omega_{X_{\mathfrak{p}^0}} \quad \left(\text{resp. } \Omega_{X'_{\mathfrak{p}^0}}, \Omega_{X_{\mathfrak{p}^1}}, \Omega_{X'_{\mathfrak{p}^1}}, \Omega_X \right)$$

instead of

$$\Omega_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}} \quad \left(\text{resp. } \Omega_{X'_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}, \Omega_{X_{\mathfrak{p}^1}/R_1}, \Omega_{X'_{\mathfrak{p}^1}/R_1}, \Omega_{X/K} \right).$$

We need to recall the following fundamental result due to Cartier (see [Kat70], Th. 7.2 for its proof):

Theorem 9. *There exists a unique homomorphism of $\mathcal{O}_{X'_{\mathfrak{p}^0}}$ -graded algebras*

$$C^{-1} : \bigoplus_{i \geq 0} \Omega_{X'_{\mathfrak{p}^0}}^i \rightarrow \bigoplus_{i \geq 0} H^i \left(F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{\mathfrak{p}^0}}^\bullet \right)$$

such that $C^{-1}d(x \otimes 1) = \text{class of } x^{p-1}dx$ for all global sections x of \mathcal{O}_X . Furthermore C^{-1} is an isomorphism and its inverse is called the Cartier isomorphism.

Observe now that since U is normal, then \mathcal{A} is projective over U (cf. Th.XI 1.4 in [Ray77]). Therefore there exists a U -very ample line bundle L on \mathcal{X} . For any $\mathfrak{p} \in U$ different from the generic point ξ , let us denote by $L_{\mathfrak{p}}$ the inverse image of L on $X_{\mathfrak{p}^0}$. Similarly we denote by L_{ξ} the inverse image of L on X . From now on, for any vector bundle $G_{\mathfrak{p}}$ over $X_{\mathfrak{p}^0}$, we will write $\deg(G_{\mathfrak{p}})$ for the degree of $G_{\mathfrak{p}}$ with respect to $L_{\mathfrak{p}}$. Analogously, if G_{ξ} is a vector bundle over X , we will write $\deg(G_{\xi})$ for the degree of G_{ξ} with respect to L_{ξ} . Now consider the vector bundle $\Omega_{\mathcal{X}/U}$ over \mathcal{X} . The map from U to \mathbb{Z} defined by

$$\mathfrak{p} \mapsto \chi((\Omega_{\mathcal{X}/U} \otimes L^m)_{\mathfrak{p}}) = \chi(\Omega_{X_{\mathfrak{p}^0}} \otimes L_{\mathfrak{p}}^m)$$

and

$$\xi \mapsto \chi((\Omega_{X/U} \otimes L^m)_\xi) = \chi(\Omega_X \otimes L_\xi^m)$$

(here χ refers to the Euler characteristic) is locally constant on U (cf. Ch.II, Sec.5 in [MRM74]). But $\deg(\Omega_{X_{\mathfrak{p}^0}})$ only depends on the polynomial map $\chi(\Omega_{X_{\mathfrak{p}^0}} \otimes L_{\mathfrak{p}}^m)$, so for every $\mathfrak{p} \in U$ we have $\deg(\Omega_{X_{\mathfrak{p}^0}}) = \deg(\Omega_X)$.

Lemma 10. *Let K , A and X be as fixed in Notations, let $\text{Stab}_A(X)$ be trivial and let n be the dimension of X over K . Then*

$$\text{Hom}_{X_{\mathfrak{p}^0}}(F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}}, \Omega_{X_{\mathfrak{p}^0}}) = 0$$

for any $k \geq 1$ and any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Let us fix $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$. We know that if

$$\mu_{\min}(F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}}) > \mu_{\max}(\Omega_{X_{\mathfrak{p}^0}})$$

then $\text{Hom}_{X_{\mathfrak{p}^0}}(F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}}, \Omega_{X_{\mathfrak{p}^0}}) = 0$. Since $\mu_{\min} \geq \bar{\mu}_{\min}$ and $\bar{\mu}_{\max} \geq \mu_{\max}$, it is sufficient to show that

$$\bar{\mu}_{\min}(F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}}) > \bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}^0}})$$

for every $k \geq 1$.

Recall now that $\text{Stab}_{A_{\mathfrak{p}^0}}(X_{\mathfrak{p}^0}) = 0$ implies $H^0(X_{\mathfrak{p}^0}, \Omega_{X_{\mathfrak{p}^0}}^\vee) = 0$ (see Appendix for this). The existence of the short exact sequence

$$0 \rightarrow \Omega_{X_{\mathfrak{p}^0}}^\vee \rightarrow i^* \Omega_{A_{\mathfrak{p}^0}}^\vee \rightarrow N \rightarrow 0$$

(here N is the normal bundle to $X_{\mathfrak{p}^0}$ in $A_{\mathfrak{p}^0}$ and $i : X_{\mathfrak{p}^0} \hookrightarrow A_{\mathfrak{p}^0}$ is the closed immersion) assure that we can apply Lemma 4 with $Y = X_{\mathfrak{p}^0}$ and $V = \Omega_{X_{\mathfrak{p}^0}}^\vee$. Therefore for any surjective finite morphism $\phi : Y' \rightarrow X_{\mathfrak{p}^0}$, we have

$$H^0(Y', \phi^* \Omega_{X_{\mathfrak{p}^0}}^\vee) \simeq H^0(X_{\mathfrak{p}^0}, \Omega_{X_{\mathfrak{p}^0}}^\vee) = 0.$$

This tells us that the first hypothesis of Lemma 7 is satisfied (again in the case $Y = X_{\mathfrak{p}^0}$ and $V = \Omega_{X_{\mathfrak{p}^0}}^\vee$). The second hypothesis is also satisfied: to see that $\Omega_{X_{\mathfrak{p}^0}}$ is globally generated it is enough to dualize the exact sequence above

$$0 \rightarrow N^\vee \rightarrow i^* \Omega_{A_{\mathfrak{p}^0}} \rightarrow \Omega_{X_{\mathfrak{p}^0}} \rightarrow 0$$

and remember that the vector bundle $\Omega_{A_{p^0}}$ is free. Therefore Lemma 7 implies $\bar{\mu}_{\min}(\Omega_{X_{p^0}}) > 0$. Using this and the equality $\bar{\mu}_{\min}(F_{X_{p^0}}^{k,*}\Omega_{X_{p^0}}) = p^k \bar{\mu}_{\min}(\Omega_{X_{p^0}})$, then all we have to verify is that

$$p\bar{\mu}_{\min}(\Omega_{X_{p^0}}) > \bar{\mu}_{\max}(\Omega_{X_{p^0}}).$$

Theorem 6 gives us the following inequality

$$\bar{\mu}_{\min}(\Omega_{X_{p^0}}) - \mu_{\min}(\Omega_{X_{p^0}}) \geq \frac{1-n}{p} \bar{\mu}_{\max}(\Omega_{X_{p^0}})$$

so that

$$p\bar{\mu}_{\min}(\Omega_{X_{p^0}}) \geq p\mu_{\min}(\Omega_{X_{p^0}}) + (1-n)\bar{\mu}_{\max}(\Omega_{X_{p^0}}).$$

We are then reduced to prove that

$$p\mu_{\min}(\Omega_{X_{p^0}}) > n\bar{\mu}_{\max}(\Omega_{X_{p^0}}).$$

We make use again of Theorem 6

$$\bar{\mu}_{\max}(\Omega_{X_{p^0}}) \leq \frac{p}{p+1-n} \mu_{\max}(\Omega_{X_{p^0}})$$

so it is enough to show that

$$(p+1-n)\mu_{\min}(\Omega_{X_{p^0}}) > n\mu_{\max}(\Omega_{X_{p^0}}).$$

If $\Omega_{X_{p^0}}$ is semistable, we obtain $p > 2n-1$. Otherwise, we can estimate $\mu_{\max}(\Omega_{X_{p^0}})$ and $\mu_{\min}(\Omega_{X_{p^0}})$ in the following way. There exists a subsheaf $0 \neq M \subsetneq \Omega_{X_{p^0}}$ such that

$$\mu_{\max}(\Omega_{X_{p^0}}) = \frac{\deg(M)}{\text{rk}(M)}$$

therefore we have $\mu_{\max}(\Omega_{X_{p^0}}) \leq \deg(M) \leq \deg(\Omega_{X_{p^0}}) - 1$. Similarly,

$$\mu_{\min}(\Omega_{X_{p^0}}) = \frac{\deg(Q)}{\text{rk}(Q)}$$

for some Q quotient of $\Omega_{X_{p^0}}$, so $\mu_{\min}(\Omega_{X_{p^0}}) \geq 1/n$. It is then sufficient to have

$$p > n^2 \deg(\Omega_{X_{p^0}}) + (n-1-n^2).$$

Since $n-1-n^2$ is always negative, we are reduced to $p > n^2 \deg(\Omega_{X_{p^0}})$. Now $\deg(\Omega_{X_{p^0}})$ is greater or equal to one, so $n^2 \deg(\Omega_{X_{p^0}}) \geq 2n-1$ for any n . This ensures us that the condition

$$p > n^2 \deg(\Omega_{X_{p^0}})$$

is sufficient to have $\mu_{\min} \left(F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}} \right) > \mu_{\max} \left(\Omega_{X_{\mathfrak{p}^0}} \right)$ for every $k \geq 1$ whether $\Omega_{X_{\mathfrak{p}^0}}$ is semistable or not. To conclude it is enough to remember that $\deg \left(\Omega_{X_{\mathfrak{p}^0}} \right)$ coincides with $\deg(\Omega_X)$. \square

Corollary 11. *The map*

$$H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}^\vee \right) \rightarrow H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}}^\vee \right)$$

is injective for every $k \geq 1$ and every $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Lemma 10 and Spziro, Lewin-Ménégaux result (stated at the end of the previous section) imply that

$$H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^{h,*} \Omega_{X_{\mathfrak{p}^0}}^\vee \right) \rightarrow H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^{h+1,*} \Omega_{X_{\mathfrak{p}^0}}^\vee \right)$$

is injective for every $h \geq 0$. Therefore the composition

$$H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}^\vee \right) \hookrightarrow H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^{2,*} \Omega_{X_{\mathfrak{p}^0}}^\vee \right) \hookrightarrow \dots \hookrightarrow H^1 \left(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^{k,*} \Omega_{X_{\mathfrak{p}^0}}^\vee \right)$$

is an injective map. \square

We are now ready to prove Theorem 12.

Theorem 12. *With the same hypotheses as in Lemma 10, for any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$, the set*

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

Notice that in [Rös13], D. Rössler proved a result on the sparsity of highly p -divisible unramified liftings implying that, for m big enough, the set

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } p^m A_{\mathfrak{p}^m}(R_m) \cap X_{\mathfrak{p}^m}(R_m) \right\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$ (cf. Th. 4.1 in [Rös13]). Theorem 12 can be viewed as an effective form of Rössler's result.

Proof. Since $\text{Crit}^1(\mathcal{X}, \mathcal{A})(\overline{k(\mathfrak{p})}) = pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)$, we have that

$$\left\{ P \in X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})}) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \right\}$$

coincides with the image of $\text{Crit}^1(\mathcal{X}, \mathcal{A})(\overline{k(\mathfrak{p})}) \rightarrow X_{\mathfrak{p}^0}(\overline{k(\mathfrak{p})})$. Therefore, the thesis of our theorem is equivalent to: $\text{Exc}^1(\mathcal{X}, \mathcal{A})$ does not coincide with $X_{\mathfrak{p}^0}$.

Let us suppose by contradiction that $\text{Exc}^1(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$. Consider the commutative diagram of $\overline{k(\mathfrak{p})}$ -schemes

$$\begin{array}{ccc} \text{Crit}^1(\mathcal{X}, \mathcal{A}) & \longrightarrow & X_{\mathfrak{p}^0} \\ \downarrow & & \downarrow \text{Id} \\ \text{Gr}_1(X_{\mathfrak{p}^1}) & \longrightarrow & X_{\mathfrak{p}^0} \end{array}$$

where the left vertical morphism is a closed immersion. Since we are assuming $\text{Exc}^1(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$, then $\text{Crit}^1(\mathcal{X}, \mathcal{A}) \rightarrow X_{\mathfrak{p}^0}$ is surjective. We choose an irreducible component

$$\text{Crit}^1(\mathcal{X}, \mathcal{A})_0 \hookrightarrow \text{Crit}^1(\mathcal{X}, \mathcal{A})$$

which dominates $X_{\mathfrak{p}^0}$. Now consider the $V\left(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}\right)$ -torsor $\pi_1 : \text{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$. Lemma 10 allows us to apply Lemma 8 with $T = \text{Gr}_1(X_{\mathfrak{p}^1})$, $Y = X_{\mathfrak{p}^0}$, $n_0 = 1$ and ϕ equal to

$$\text{Crit}^1(\mathcal{X}, \mathcal{A})_0 \rightarrow X_{\mathfrak{p}^0}.$$

We have that $\phi^* \text{Gr}_1(X_{\mathfrak{p}^1})$ is trivial as $V\left(\phi^* F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}\right)$ -torsor, since

$$\text{Crit}^1(\mathcal{X}, \mathcal{A})_0 \subseteq \text{Gr}_1(X_{\mathfrak{p}^1}).$$

Hence we obtain that $\pi_1 : \text{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$ is trivial as $V\left(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}\right)$ -torsor. Let us take a section $\sigma : X_{\mathfrak{p}^0} \rightarrow \text{Gr}_1(X_{\mathfrak{p}^1})$. By definition of Greenberg transform, the map σ over $\overline{k(\mathfrak{p})}$ corresponds to a map $\bar{\sigma} : \mathbb{W}_1(X_{\mathfrak{p}^0}) \rightarrow X_{\mathfrak{p}^1}$ over R_1 . We can precompose $\bar{\sigma}$ with the morphism $t : X_{\mathfrak{p}^1} \rightarrow \mathbb{W}_1(X_{\mathfrak{p}^0})$ corresponding to

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathfrak{p}^0}}) &\rightarrow \mathcal{O}_{X_{\mathfrak{p}^1}} \\ (a_0, a_1) &\mapsto \tilde{a}_0^p + \tilde{a}_1 p \end{aligned}$$

where \tilde{a}_i lifts a_i . Consider now the following diagram

$$\begin{array}{ccccc} X_{\mathfrak{p}^1} & \xrightarrow{t} & \mathbb{W}_1(X_{\mathfrak{p}^0}) & \xrightarrow{\bar{\sigma}} & X_{\mathfrak{p}^1} \\ \uparrow & & \uparrow & & \uparrow \\ X_{\mathfrak{p}^0} & \xrightarrow{F_{X_{\mathfrak{p}^0}}} & X_{\mathfrak{p}^0} & \xrightarrow{\text{Id}} & X_{\mathfrak{p}^0} \end{array}$$

Notice that the composition

$$X_{\mathfrak{p}^0} \longrightarrow X_{\mathfrak{p}^1} \xrightarrow{t} \mathbb{W}_1(X_{\mathfrak{p}^0})$$

simply corresponds to the map

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathfrak{p}^0}}) &\rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \\ (a_0, a_1) &\mapsto \tilde{a}_0^p. \end{aligned}$$

Therefore the left square in the above diagram is commutative. The properties of the Greenberg transform and the equality $\pi_1 \circ \sigma = \text{Id}_{X_{\mathfrak{p}^0}}$ imply that the right square is commutative too. We obtain in this way that $\bar{\sigma} \circ t : X_{\mathfrak{p}^1} \rightarrow X_{\mathfrak{p}^1}$ is a lift of the Frobenius $F_{X_{\mathfrak{p}^0}}$ over R_1 .

The diagram below is also commutative

$$\begin{array}{ccccc} X_{\mathfrak{p}^1} & \xrightarrow{t} & W_1(X_{\mathfrak{p}^0}) & \xrightarrow{\bar{\sigma}} & X_{\mathfrak{p}^1} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(R_1) & \xrightarrow{F_{R_1}} & \text{Spec}(R_1) & \xrightarrow{\text{Id}} & \text{Spec}(R_1) \end{array}$$

In fact, by definition, $\bar{\sigma}$ is a morphism over R_1 , so the right square is commutative. The commutativity of the left square is easy to check, since we know explicitly t and F_{R_1} .

The commutativity of the diagram above implies the existence of a morphism

$$\tilde{F} : X_{\mathfrak{p}^1} \rightarrow X'_{\mathfrak{p}^1}$$

over R_1 lifting $F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}$. Now we use a classical argument involving lifting the Frobenius and the Cartier isomorphism (cf. part (b) of the proof of Théorème 2.1 in [DI87]). Since

$$F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}^* : \Omega_{X'_{\mathfrak{p}^0}} \rightarrow F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{\mathfrak{p}^0}}$$

is the zero map, then the image of $\tilde{F}^* : \Omega_{X'_{\mathfrak{p}^1}} \rightarrow \tilde{F}_* \Omega_{X_{\mathfrak{p}^1}}$ is contained in $p\tilde{F}_* \Omega_{X_{\mathfrak{p}^1}}$. Furthermore, the multiplication by p induces an isomorphism

$$p : F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{\mathfrak{p}^0}} \rightarrow p\tilde{F}_* \Omega_{X_{\mathfrak{p}^1}},$$

so that there exists a unique map

$$f := p^{-1}\tilde{F}^* : \Omega_{X'_{\mathfrak{p}^0}} \rightarrow F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})},*} \Omega_{X_{\mathfrak{p}^0}}$$

making the diagram below commutative

$$\begin{array}{ccc}
\Omega_{X'_{\mathfrak{p}1}} & \xrightarrow{\tilde{F}^*} & p\tilde{F}_*\Omega_{X_{\mathfrak{p}1}} \\
\downarrow & & \uparrow p \\
\Omega_{X'_{\mathfrak{p}0}} & \xrightarrow{f} & F_{X_{\mathfrak{p}0}/\overline{k(\mathfrak{p})},*}\Omega_{X_{\mathfrak{p}0}}
\end{array}$$

If x is a local section of $\mathcal{O}_{X_{\mathfrak{p}1}}$ with reduction x_0 modulo p , then

$$\tilde{F}^*(x \otimes 1) = x^p + pu(x)$$

where $u(x)$ is a local section of $\mathcal{O}_{X_{\mathfrak{p}1}}$ and

$$f(dx_0 \otimes 1) = x_0^{p-1}dx_0 + d(u(x)).$$

In particular we have that $df = 0$ and the map

$$H^1 \circ f : \Omega_{X'_{\mathfrak{p}0}} \rightarrow F_{X_{\mathfrak{p}0}/\overline{k(\mathfrak{p})},*}\Omega_{X_{\mathfrak{p}0}} \rightarrow H^1 \left(F_{X_{\mathfrak{p}0}/\overline{k(\mathfrak{p})},*}\Omega_{X_{\mathfrak{p}0}}^\bullet \right)$$

coincides with the inverse of the Cartier isomorphism C^{-1} in degree one (see Theorem 9). Therefore $H^1 \circ f$ is an isomorphism which implies that f is not the zero map. We can now consider the adjoint of f

$$\bar{f} : F_{X_{\mathfrak{p}0}}^*\Omega_{X_{\mathfrak{p}0}} = F_{X_{\mathfrak{p}0}/\overline{k(\mathfrak{p})}}^*\Omega_{X'_{\mathfrak{p}0}} \rightarrow \Omega_{X_{\mathfrak{p}0}}.$$

Being f nonzero, then also \bar{f} is nonzero and this contradicts Lemma 10. \square

6. THE MANIN-MUMFORD CONJECTURE FOR SUBVARIETIES WITH AMPLE COTANGENT BUNDLE

Let K, X, A be as fixed in Notations and let $\text{Stab}_A(X)$ be trivial. In this section we also suppose that Ω_X is ample. Then by Proposition 4.4 in [Har66] we know that $\Omega_{X_{\mathfrak{p}0}}$ is ample for all \mathfrak{p} in a nonempty open subscheme W of U . Let us denote by \overline{U} the open subscheme

$$\overline{U} := W \cap \tilde{U} \subseteq U,$$

so that every $\mathfrak{p} \in \overline{U}$ verifies:

- K/\mathbb{Q} is unramified at \mathfrak{p} ,
- X/K has good reduction at \mathfrak{p} ,
- $\text{Stab}_{A_{\mathfrak{p}0}}(X_{\mathfrak{p}0})$ is trivial,
- $\Omega_{X_{\mathfrak{p}0}}$ is ample.

Theorem 13. *Let K , A and X be as fixed in Notations, let $\text{Stab}_A(X)$ be trivial, let Ω_X be ample and n be the dimension of X over K . For any $\mathfrak{p} \in \overline{U}$ above a prime $p > n^2 \deg(\Omega_X)$ the set*

$$pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)$$

is finite.

Proof. This is clearly equivalent to show that the scheme

$$\text{Crit}^1(\mathcal{X}, \mathcal{A}) = [p]_* \text{Gr}_1(A_{\mathfrak{p}^1}) \cap \text{Gr}_1(X_{\mathfrak{p}^1})$$

is finite over $\overline{k(\mathfrak{p})}$. We have already observed that $[p]_* \text{Gr}_1(A_{\mathfrak{p}^1})$ is proper over $\overline{k(\mathfrak{p})}$, being the greatest abelian subvariety of $\text{Gr}_1(A_{\mathfrak{p}^1})$. Therefore if we prove that $\text{Gr}_1(X_{\mathfrak{p}^1})$ is affine we are done, since any affine proper morphism is finite. To prove that $\text{Gr}_1(X_{\mathfrak{p}^1})$ is affine, we use exactly the same argument the reader can find in Proposition 1.10 in [Bui96].

Let us consider the $V(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}})$ -torsor $\pi_1 : \text{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$. While proving Theorem 12, we have seen that π_1 is not trivial. Now let

$$\eta \in H^1(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}^\vee)$$

be the class defined by π_1 . Under the natural isomorphism

$$H^1(X_{\mathfrak{p}^0}, F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}^\vee) \simeq \text{Ext}^1(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}, \mathcal{O}_{X_{\mathfrak{p}^0}})$$

η corresponds to some extension

$$(1) \quad 0 \rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \rightarrow E \rightarrow F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}} \rightarrow 0.$$

We denote by $\mathbb{P}(E)$ the projective bundle over $X_{\mathfrak{p}^0}$ associated to E , and by D the divisor $\mathbb{P}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) \subseteq \mathbb{P}(E)$. Then the torsor π_1 identifies with $\mathbb{P}(E) \setminus D$.

The fact that π_1 is not the trivial torsor implies that the short sequence (1) is nonsplit. We can say even more: for any proper surjective morphism $\phi : Y' \rightarrow X_{\mathfrak{p}^0}$ with Y' irreducible, the pullback of (1) through ϕ is nonsplit. This is exactly what Lemma 8 tells us (notice that we can apply Lemma 8 to our situation thanks to Lemma 10). This and our assumption on the ampleness of $\Omega_{X_{\mathfrak{p}^0}}$ allow us to apply Corollary 2 in Sec. 1 of [MD84]: we obtain that E is ample, i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample. But D belongs to the linear system of $\mathcal{O}_{\mathbb{P}(E)}(1)$, hence D is ample and $\text{Gr}_1(X_{\mathfrak{p}^1})$ is affine. \square

It is now easy to deduce a new proof of the Manin-Mumford conjecture in the case of a subvariety with ample cotangent bundle.

Theorem 14. *Let K , A and X be as fixed in Notations, let $\text{Stab}_A(X)$ be trivial and let Ω_X be ample. The set*

$$\text{Tor}(A(\overline{K})) \cap X(\overline{K})$$

is finite.

Proof. Let us fix $p > n^2 \deg(\Omega_X)$ and write

$$\text{Tor}(A(\overline{K})) = \text{Tor}^p(A(\overline{K})) \oplus \text{Tor}_p(A(\overline{K})).$$

We prove the finiteness of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ and the finiteness of $\text{Tor}_p(A(\overline{K})) \cap X(\overline{K})$ separately.

If $\mathfrak{p} \in \overline{U}$ is above p , then the field of definition of every prime-to- p torsion point is unramified at \mathfrak{p} . This implies

$$\text{Tor}^p(A(\overline{K})) \cap X(\overline{K}) \subseteq \text{Tor}^p(\mathcal{A}(R)) \cap \mathcal{X}(R)$$

where R is the ring of Witt vectors with coordinates in $\overline{k(\mathfrak{p})}$. By precomposing with the morphism $\text{Spec } R_1 \rightarrow \text{Spec } R$ we obtain a homomorphism $\delta : \mathcal{A}(R) \rightarrow A_{\mathfrak{p}^1}(R_1)$. The restriction of δ to $\text{Tor}^p(\mathcal{A}(R))$ is injective: in fact if $p \geq 3$ then the restriction to the entire torsion is injective and if $p = 2$ the only torsion points contained in the kernel are of order 2 (see [[Sil86], Chapter IV, Theorem 6.1] for this). So we have an injection

$$\text{Tor}^p(\mathcal{A}(R)) \cap \mathcal{X}(R) \hookrightarrow \text{Tor}^p(A_{\mathfrak{p}^1}(R_1)) \cap X_{\mathfrak{p}^1}(R_1).$$

Since $\text{Tor}^p(A_{\mathfrak{p}^1}(R_1)) \subseteq pA_{\mathfrak{p}^1}(R_1)$, we obtain

$$\text{Tor}^p(A(\overline{K})) \cap X(\overline{K}) \subseteq pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1).$$

Therefore $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ is a finite set by Theorem 13.

Now, for brevity's sake, let us write $Z := \text{Tor}_p(A(\overline{K})) \cap X(\overline{K})$ and let us denote by \overline{Z} the Zariski closure of Z . We use here an argument given by D. Rössler in the last remark of [Rös05]. Let us write $A[p]$ for the p -torsion points of A and K' for the extension of K generated by the points in $A[p]$. Then Z is the union

$$Z = (Z \cap A(K')) \bigcup (Z \cap A(K')^c).$$

Since Z is Zariski dense in \overline{Z} , then at least one of the above sets has to be dense in Z . But $(Z \cap A(K')^c)$ is not dense in \overline{Z} : this is a consequence of Proposition 3 in [Rös05] and of our assumption $\text{Stab}(X) = 0$. Then $(Z \cap A(K'))$, which is a finite set by the theorem of Mordell-Weil, is dense in \overline{Z} . This implies that \overline{Z} is finite. In particular Z is finite. \square

7. AN EXPLICIT BOUND FOR THE CARDINALITY OF THE PRIME-TO- p TORSION OF DEBARRE'S SUBVARIETIES

In this section we provide an upper bound for the number of points in the set $\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ in the case in which X is a subvariety with ample cotangent bundle of the type studied by Debarre in [Deb05].

Let A/K be an abelian variety of dimension n and let $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c \in \mathbb{N}$ be greater than $n/2$ and $e \in \mathbb{N}$. For $H_1, H_2, \dots, H_c \in |L^e|$ general and e sufficiently big, the subvariety $X := H_1 \cap H_2 \cap \dots \cap H_c$ has ample cotangent bundle (by Theorem 8 in [Deb05]). Suppose that X is smooth and has trivial stabilizer.

First of all, let us take a sufficiently small open $V \subseteq \mathrm{Spec}(\mathcal{O}_K)$ such that A extends over V to an abelian scheme \mathcal{A} , L extends to a V -very ample line bundle \mathcal{L} , H_i extends to \mathcal{H}_i for every i and $\mathcal{X} := \mathcal{H}_1 \cap \mathcal{H}_2 \cap \dots \cap \mathcal{H}_c$ is smooth and has ample cotangent bundle. We can restrict V if necessary and suppose that for all $\mathfrak{p} \in V$ the stabilizer $\mathrm{Stab}_{A_{\mathfrak{p}^0}}(X_{\mathfrak{p}^0})$ is trivial and K/\mathbb{Q} is unramified at \mathfrak{p} .

For any $m \in \mathbb{N}$ we define

$$C^m := \left\{ (r_1, r_2, \dots) \mid r_i \in \mathbb{N} \text{ and } \sum i r_i = m \right\}$$

and for any $\underline{\beta} \in C^m$ we define

$$M_{\underline{\beta}} := (-1)^{\sum \beta_i} \binom{\sum \beta_i}{\beta_1, \beta_2, \dots}$$

$$R_{\underline{\beta}}^c := \prod_{j \geq 1} \binom{c}{j}^{\beta_j}.$$

Finally for any $m \in \mathbb{N}$ we write

$$W_{m,c} := \sum_{\underline{\beta} \in C^m} M_{\underline{\beta}} R_{\underline{\beta}}^c.$$

Theorem 15. *Let K be a number field, A/K be an abelian variety of dimension n and let $L = \mathcal{O}_A(D)$ be a very ample line bundle on A . Let $c, e \in \mathbb{N}$ with $c \geq n/2$. Let $H_1, H_2, \dots, H_c \in |L^e|$ be general and e be sufficiently big, so that the subvariety $X := H_1 \cap H_2 \cap \dots \cap H_c$ has ample cotangent bundle. Suppose that X is smooth and has trivial stabilizer. If \mathfrak{p} is in V (the open subscheme of $\mathrm{Spec}(\mathcal{O}_K)$ defined above) and \mathfrak{p} is above a prime $p > (n - c)^2 c e^{c+1} (L^n)$, then the cardinality of $\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K})$*

is bounded by

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) (L^n)^2.$$

Proof. Fix \mathfrak{p} above a prime $p > (n-c)^2 \deg_{L|X} \Omega_X$. Then we know that

$$\mathrm{Tor}^p(A(\overline{K})) \cap X(\overline{K}) \subseteq ([p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \cap \mathrm{Gr}_1(X_{\mathfrak{p}^1})) (\overline{k(\mathfrak{p})})$$

which is finite by Theorem 13. We have two exact sequences of vector bundles

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \rightarrow E_X \rightarrow F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}} \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_{A_{\mathfrak{p}^0}} \rightarrow E_A \rightarrow F_{A_{\mathfrak{p}^0}}^* \Omega_{A_{\mathfrak{p}^0}} \rightarrow 0 \end{aligned}$$

corresponding to the torsors $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \rightarrow X_{\mathfrak{p}^0}$ and $\mathrm{Gr}_1(A_{\mathfrak{p}^1}) \rightarrow A_{\mathfrak{p}^0}$. If we denote by D_X the divisor $\mathbb{P}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) \subseteq \mathbb{P}(E_X)$ and D_A the divisor $\mathbb{P}(F_{A_{\mathfrak{p}^0}}^* \Omega_{A_{\mathfrak{p}^0}}) \subseteq \mathbb{P}(E_A)$, then we have two identifications

$$\begin{aligned} \mathrm{Gr}_1(X_{\mathfrak{p}^1}) &\simeq \mathbb{P}(E_X) \setminus D_X \\ \mathrm{Gr}_1(A_{\mathfrak{p}^1}) &\simeq \mathbb{P}(E_A) \setminus D_A. \end{aligned}$$

We shall write i for the closed embedding $i : \Omega_{X_{\mathfrak{p}^0}} \rightarrow \Omega_{A_{\mathfrak{p}^0}}$. It is not difficult to show that there is a natural restriction homomorphism $i^* E_A \rightarrow E_X$ prolonging the homomorphism $i^* \Omega_{A_{\mathfrak{p}^0}} \rightarrow \Omega_{X_{\mathfrak{p}^0}}$. The homomorphism $i^* E_A \rightarrow E_X$ is clearly surjective, so it induces a closed embedding $j : \mathbb{P}(E_X) \hookrightarrow \mathbb{P}(E_A)$ prolonging the embedding $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \hookrightarrow \mathrm{Gr}_1(A_{\mathfrak{p}^1})$. Therefore we have a commutative diagram

$$\begin{array}{ccc}
& & [p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \\
& & \downarrow \\
\mathrm{Gr}_1(X_{\mathfrak{p}^1}) & \hookrightarrow & \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \\
\downarrow & & \downarrow \\
\mathbb{P}(E_X) & \xrightarrow{j} & \mathbb{P}(E_A) \\
\downarrow \pi_X & & \downarrow \pi_A \\
X_{\mathfrak{p}^0} & \xrightarrow{i} & A_{\mathfrak{p}^0} \\
\downarrow & \nearrow & \\
\mathrm{Spec}(\overline{k(\mathfrak{p})}) & &
\end{array}
\quad \begin{array}{l} \\ \\ \\ \\ \\ \end{array}
\quad \begin{array}{l} \\ \\ \\ \\ \curvearrowright T \\ \end{array}$$

Let us denote by $\mathcal{L}_{\mathfrak{p}}$ the base change of \mathcal{L} to $A_{\mathfrak{p}^0}$. It is standard to prove that the line bundle

$$\mathcal{H} := \pi_A^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_A)}(1).$$

is very ample on $\mathbb{P}(E_A)$ (cf. pag 4 in [BV96]). We have

$$\mathcal{H}|_{\mathbb{P}(E_X)} = \pi_X^* i^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_X)}(1)$$

$$\mathcal{H}|_{[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})} = T^* \mathcal{L}_{\mathfrak{p}}$$

since $D_A \in |\mathcal{O}_{\mathbb{P}(E_A)}(1)|$ and $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \subseteq \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_A) \setminus D_A$. We know that $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})$ is the maximal abelian subvariety of $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ and we also know that the multiplication by p map on $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ factors through the isogeny T . This implies that T has degree at most p^{2n} , so we have the following estimate

$$\deg_{\mathcal{H}}([p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})) \leq p^{2n} (\mathcal{L}_{\mathfrak{p}}^n).$$

Let us now consider $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$. It coincides with

$$(2) \quad \int_{\mathbb{P}(E_X)} c_1(\mathcal{H}|_{\mathbb{P}(E_X)})^{2n-2c}$$

where c_1 stands for the first Chern class in the Chow ring and $\int_{\mathbb{P}(E_X)}$ stands for the push-forward morphism to $\mathrm{Spec}(\overline{k(\mathfrak{p})})$ in the Chow theory. Since

$$c_1(\mathcal{H}|_{\mathbb{P}(E_X)}) = c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}}) + c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))$$

we can re-write (2) as

$$\int_{\mathbb{P}(E_X)} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}})^h \cdot c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h}.$$

Equivalently

$$\int_{X_{\mathfrak{p}^0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot \pi_{X,*} \left(c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h} \right)$$

and by definition of Segre class this is

$$\int_{X_{\mathfrak{p}^0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X).$$

But $s_k = 0$ if $k < 0$, so we end up with

$$\int_{X_{\mathfrak{p}^0}} \sum_{h=0}^{n-c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X).$$

Now the exact sequence

$$0 \rightarrow \mathcal{O}_{X_{\mathfrak{p}^0}} \rightarrow E_X \rightarrow F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}} \rightarrow 0$$

implies

$$s_{n-c-h}(E_X) = \sum_{i+j=n-c-h} s_i(\mathcal{O}_{X_{\mathfrak{p}^0}}) s_j(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) = s_{n-c-h}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}})$$

and so

$$s_{n-c-h}(E_X) = s_{n-c-h}(F_{X_{\mathfrak{p}^0}}^* \Omega_{X_{\mathfrak{p}^0}}) = p^{n-c-h} s_{n-c-h}(\Omega_{X_{\mathfrak{p}^0}})$$

(here we have used the following fact: the pullback of a cycle η of codimension j through the Frobenius map coincides with $p^j \eta$). Therefore we have to study the following sum

$$(3) \quad \sum_{h=0}^{n-c} \binom{2n-2c}{h} p^{n-c-h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(\Omega_{X_{\mathfrak{p}^0}})$$

We can rewrite $s_{n-c-h}(\Omega_{X_{\mathfrak{p}^0}})$ as a function of $c_1(i^* \mathcal{L}_{\mathfrak{p}})$. Recall that if $\sum_{m \geq 0} a_m t^m$ is a formal power series with $a_0 = 1$ then its inverse (for the multiplication) is

$$\left(\sum_{m \geq 0} a_m t^m \right)^{-1} = \sum_{m \geq 0} \left(\sum_{\underline{\beta} \in C^m} M_{\underline{\beta}} \prod_{i \geq 1} a_i^{\beta_i} \right) t^m.$$

Since

$$s_t(\Omega_{X_{\mathfrak{p}^0}}) = 1 + s_1(\Omega_{X_{\mathfrak{p}^0}})t + s_2(\Omega_{X_{\mathfrak{p}^0}})t^2 + \dots$$

is the inverse power series of

$$c_t(\Omega_{X_{\mathfrak{p}_0}}) = 1 + c_1(\Omega_{X_{\mathfrak{p}_0}})t + c_2(\Omega_{X_{\mathfrak{p}_0}})t^2 + \dots$$

we obtain

$$s_{n-c-h}(\Omega_{X_{\mathfrak{p}_0}}) = \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} c_i(\Omega_{X_{\mathfrak{p}_0}})^{\beta_i} = \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} (-1)^{i\beta_i} c_i(T_{X_{\mathfrak{p}_0}})^{\beta_i}$$

The normal exact sequence for i

$$0 \rightarrow T_{X_{\mathfrak{p}_0}} \rightarrow i^*T_{A_{\mathfrak{p}_0}} \rightarrow N \rightarrow 0$$

implies

$$c_t(T_{X_{\mathfrak{p}_0}})c_t(N) = c_t(i^*T_{A_{\mathfrak{p}_0}}) = 1$$

where N is the normal bundle for i , so we have that $c_t(T_{X_{\mathfrak{p}_0}})$ is the inverse of $c_t(N)$.

Recalling that

$$c_t(N) = (1 + c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)t)^c$$

and applying the formula for the inverse of a formal power series to $c_t(T_{X_{\mathfrak{p}_0}})$ we get

$$\begin{aligned} c_i(T_{X_{\mathfrak{p}_0}}) &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} c_j(N)^{\beta_j} = \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \left(\binom{c}{j} c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)^j \right)^{\beta_j} \\ &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \binom{c}{j}^{\beta_j} c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)^{j\beta_j} \\ &= c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)^i \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} R_{\underline{\beta}}^c \\ &= c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)^i W_{i,c}. \end{aligned}$$

Therefore we have

$$\begin{aligned} s_{n-c-h}(\Omega_{X_{\mathfrak{p}_0}}) &= \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} (-1)^{i\beta_i} c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)^{i\beta_i} W_{i,c}^{\beta_i} \\ &= \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} (-1)^{n-c-h} c_1(i^*\mathcal{L}_{\mathfrak{p}}^e)^{n-c-h} \prod_{i \geq 1} W_{i,c}^{\beta_i} \\ &= \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} (-e)^{n-c-h} c_1(i^*\mathcal{L}_{\mathfrak{p}})^{n-c-h} \prod_{i \geq 1} W_{i,c}^{\beta_i}. \end{aligned}$$

Substituting in 3, we obtain

$$\left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-pe)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) c_1(i^*\mathcal{L}_{\mathfrak{p}})^{n-c}.$$

Therefore $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-pe)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) \int_{X_{\mathfrak{p}^0}} c_1(i^* \mathcal{L}_{\mathfrak{p}})^{n-c}$$

Now since $X_{\mathfrak{p}^0} = H_{1,\mathfrak{p}} \cap \dots \cap H_{c,\mathfrak{p}}$ where $H_{1,\mathfrak{p}}, \dots, H_{c,\mathfrak{p}}$ belong to $|\mathcal{L}_{\mathfrak{p}}^e|$, we have

$$\int_{X_{\mathfrak{p}^0}} c_1(i^* \mathcal{L}_{\mathfrak{p}})^{n-c} = e^c \int_{A_{\mathfrak{p}^0}} c_1(\mathcal{L}_{\mathfrak{p}})^n = e^c (\mathcal{L}_{\mathfrak{p}}^n)$$

and

$$\deg_{\mathcal{H}}(\mathbb{P}(E_X)) = \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) (\mathcal{L}_{\mathfrak{p}}^n).$$

Now Bézout's theorem in Fulton's form (cf. [Ful97], p. 148) says that the number of irreducible components in the intersection of two projective varieties of degrees d_1, d_2 cannot exceed $d_1 d_2$. In particular

$$\#(\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})) \leq p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} e^{n-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} W_{i,c}^{\beta_i} \right) (\mathcal{L}_{\mathfrak{p}}^n)^2.$$

Notice that $(\mathcal{L}_{\mathfrak{p}}^n) = (L^n)$, by the same reasoning done after Theorem 9. To conclude we need to compute $\deg_{L|X} \Omega_X$. We have seen that

$$c_1(\Omega_X) = -c_1(T_X) = -c_1(L|_X^e) W_{1,c}$$

and $W_{1,c} = -c$, so $c_1(\Omega_X) = c e c_1(L|_X)$ and

$$\begin{aligned} \deg_{L|X} \Omega_X &= \int_X c_1(\Omega_X) c_1(L|_X)^{n-c-1} = c e \int_X c_1(L|_X)^{n-c} \\ &= c e \int_A c_1(L)^{n-c} c_1(L^e)^c \\ &= c e^{c+1} \int_A c_1(L)^n \\ &= c e^{c+1} (L^n) \end{aligned}$$

□

Remark 16. (1) We would like to point out that in the case of a simple abelian variety, Theorem 15 can be refined substituting “ e be sufficiently big” with “ e be strictly bigger than n ” (cf. Thm 7, [Deb05]). Analogously, in the case of an abelian variety of dimension 4, we can substitute “ e be sufficiently big” with “ e strictly bigger than 4” (cf. Thm 9, [Deb05]).

- (2) *The class of subvarieties with ample cotangent bundle provided by Debarre is bigger than the one we have considered, i.e. one can take any intersection $X := H_1 \cap H_2 \cap \dots \cap H_c$ where $H_i \in |L_i^{e_i}|$ for some very ample line bundles L_i and some e_i big enough. It is easy to make our proof work in the case $L_1 = L_2 = \dots = L_c$ and e_1, e_2, \dots, e_c arbitrary (and big enough): we have*

$$c_j(N) = \sum_{1 \leq i_1 < \dots < i_j \leq c} e_{i_1} \cdots e_{i_j} c_1(i^* \mathcal{L}_{\mathbf{p}})^j$$

which implies

$$(4) \quad c_i(T_{X_{\mathbf{p}^0}}) = c_1(i^* \mathcal{L}_{\mathbf{p}})^i Z_{i,c,\underline{e}}$$

where $\underline{e} = (e_1, \dots, e_c)$ and

$$Z_{i,c,\underline{e}} := \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \left(\sum_{1 \leq i_1 < \dots < i_j \leq c} e_{i_1} \cdots e_{i_j} \right)^{\beta_j}.$$

Therefore $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} Z_{i,c,\underline{e}}^{\beta_i} \right) \int_{X_{\mathbf{p}^0}} c_1(i^* \mathcal{L}_{\mathbf{p}})^{n-c}.$$

Since

$$\begin{aligned} \int_{X_{\mathbf{p}^0}} c_1(i^* \mathcal{L}_{\mathbf{p}})^{n-c} &= \int_{A_{\mathbf{p}^0}} c_1(\mathcal{L}_{\mathbf{p}})^{n-c} c_1(\mathcal{L}_{\mathbf{p}}^{e_1}) \cdots c_1(\mathcal{L}_{\mathbf{p}}^{e_c}) \\ &= \left(\prod_{i=1}^c e_i \right) \int_{A_{\mathbf{p}^0}} c_1(\mathcal{L}_{\mathbf{p}})^n = \left(\prod_{i=1}^c e_i \right) (\mathcal{L}_{\mathbf{p}}^n) \end{aligned}$$

we get the following bound for the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$:

$$p^{2n} \left(\sum_{h=0}^{n-c} \binom{2n-2c}{h} (-p)^{n-c-h} \cdot \sum_{\underline{\beta} \in C^{n-c-h}} M_{\underline{\beta}} \prod_{i \geq 1} Z_{i,c,\underline{e}}^{\beta_i} \right) \left(\prod_{i=1}^c e_i \right) (\mathcal{L}_{\mathbf{p}}^n)^2.$$

This bound holds for $p > (n-c)^2 \deg_{L|_X} \Omega_X$ such that there exists $\mathbf{p} \in V$ above p . We compute

$$c_1(\Omega_X) = -c_1(T_X) = -c_1(L|_X) Z_{1,c,\underline{e}}$$

and $Z_{1,c,\underline{e}} = -\sum_{i=1}^c e_i$, so

$$\begin{aligned} \deg_{L|X} \Omega_X &= \int_X c_1(\Omega_X) c_1(L|X)^{n-c-1} \\ &= \left(\sum_{i=1}^c e_i \right) \int_X c_1(L|X)^{n-c} \\ &= \left(\sum_{i=1}^c e_i \right) \left(\prod_{i=1}^c e_i \right) (L^n). \end{aligned}$$

(3) One could be tempted to generalize Theorem 15 also to the case in which H_1, \dots, H_c belong to different $|L_1^{e_1}|, \dots, |L_c^{e_c}|$. In this situation one would have

$$c_t(N) = \prod_{i=1}^c \left(1 + c_1(i^* \mathcal{L}_{i,p}^{e_i}) t \right)$$

and

$$c_j(N) = \sum_{1 \leq i_1 < \dots < i_j \leq c} \prod_{k=i_1}^{i_j} c_1(i^* \mathcal{L}_{k,p}^{e_k}).$$

Therefore

$$\begin{aligned} c_i(T_{X_{p_0}}) &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} c_j(N)^{\beta_j} \\ &= \sum_{\underline{\beta} \in C^i} M_{\underline{\beta}} \prod_{j \geq 1} \left(\sum_{1 \leq i_1 < \dots < i_j \leq c} \prod_{k=i_1}^{i_j} c_1(i^* \mathcal{L}_{k,p}^{e_k}) \right)^{\beta_j}. \end{aligned}$$

This expression for $c_i(T_{X_{p_0}})$ is more complex than the corresponding one in (4) in the case $L_1 = L_2 = \dots = L_c$. For this reason, a nice formula for the cardinality of $\text{Tor}^p(A(\overline{K})) \cap X(\overline{K})$ does not seem easily attainable in this general context.

8. APPENDIX

The following fact is well-known, but we were not able to find a published proof of it, so we give one here.

Fact 17. *Let A be an abelian variety over an algebraically closed field k and let X be a closed integral subscheme, smooth over k . If $\text{Stab}_A(X)$ is trivial, then $H^0(X, \Omega_{X/k}^\vee) = 0$.*

Proof. Let $k(\varepsilon) := k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers and let $A_{k(\varepsilon)}$ (resp. $X_{k(\varepsilon)}$) be the base change of A (resp. X) to $k(\varepsilon)$. Notice that the base change of $A_{k(\varepsilon)}$ (resp. $X_{k(\varepsilon)}$) to k gives A (resp. X). This means that we have two closed immersions $j_A : A \hookrightarrow A_{k(\varepsilon)}$ and $j_X : X \hookrightarrow X_{k(\varepsilon)}$. As usual, we shall say that $h : A_{k(\varepsilon)} \rightarrow A_{k(\varepsilon)}$ (resp. $f : X_{k(\varepsilon)} \rightarrow X_{k(\varepsilon)}$) is a deformation of the identity on A (resp. on X) if

$$h_0 := h \times \text{Id}_k : A_{k(\varepsilon)} \times_{k(\varepsilon)} k \rightarrow A_{k(\varepsilon)} \times_{k(\varepsilon)} k$$

$$(\text{resp. } f_0 := f \times \text{Id}_k : X_{k(\varepsilon)} \times_{k(\varepsilon)} k \rightarrow X_{k(\varepsilon)} \times_{k(\varepsilon)} k)$$

is the identity on A (resp. on X). The set of deformations of the identity on A (resp. on X) is nonempty, since it contains the identity on $A_{k(\varepsilon)}$ (resp. on $X_{k(\varepsilon)}$). Theorem 8.5.9 in [FIK⁺] tells us that the deformations of the identity on A form an affine space under

$$H^0\left(A, \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} j_A^* \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee\right).$$

Here the sheaf of ideals $\varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \subseteq \mathcal{O}_{A_{k(\varepsilon)}}$ is seen as a \mathcal{O}_A -module thanks to the fact $\varepsilon^2 = 0$. We have

$$\begin{aligned} \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} j_A^* \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee &\simeq \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} \left(\mathcal{O}_A \otimes_{\mathcal{O}_{A_{k(\varepsilon)}}} \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee \right) \\ &\simeq \varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_{A_{k(\varepsilon)}}} \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee. \end{aligned}$$

Now notice that the ideal $(\varepsilon) \subseteq k(\varepsilon)$ is isomorphic to k as a $k(\varepsilon)$ -algebra. Tensoring with $\mathcal{O}_{A_{k(\varepsilon)}}$, we obtain that $\varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \simeq \mathcal{O}_A$ as $\mathcal{O}_{A_{k(\varepsilon)}}$ -algebras. This implies

$$\varepsilon \mathcal{O}_{A_{k(\varepsilon)}} \otimes_{\mathcal{O}_A} j_A^* \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee \simeq \mathcal{O}_A \otimes_{\mathcal{O}_{A_{k(\varepsilon)}}} \Omega_{A_{k(\varepsilon)}/k(\varepsilon)}^\vee \simeq \Omega_{A/k}^\vee.$$

Therefore, the deformations of the identity on A form an affine space under $H^0\left(A, \Omega_{A/k}^\vee\right)$. Similarly, the deformations of the identity on X form an affine space under $H^0\left(X, \Omega_{X/k}^\vee\right)$. Let us consider now the exact sequence

$$0 \rightarrow \Omega_{X/k}^\vee \rightarrow i^* \Omega_{A/k}^\vee \rightarrow N \rightarrow 0$$

where $i : X \hookrightarrow A$ is the closed immersion and N is the normal bundle to X in A . On the global sections we obtain the exact sequence

$$0 \rightarrow H^0\left(X, \Omega_{X/k}^\vee\right) \rightarrow H^0\left(X, i^* \Omega_{A/k}^\vee\right) \rightarrow H^0(X, N).$$

Since $\Omega_{A/k}^\vee \simeq \mathcal{O}_A^{\oplus \dim A}$, we have that $i^* \Omega_{A/k}^\vee \simeq \mathcal{O}_X^{\oplus \dim A}$. This and the irreducibility of X imply that we have an isomorphism

$$H^0\left(X, i^* \Omega_{A/k}^\vee\right) \simeq H^0\left(A, \Omega_{A/k}^\vee\right).$$

We deduce that the set of deformations of the identity on X is a subset of the set of deformations of the identity on A : a deformation of the identity on A comes from a deformation of the identity on X if and only if it corresponds to a vector field of A tangent to X , i.e. there is a bijection between the set of deformations of the identity on A coming from deformations of the identity on X and $H^0(X, \Omega_{X/k}^\vee)$.

Now let us consider the closed subscheme $\text{Stab}_A(X)$. It represents the functor that associates to any S scheme over k the set

$$\{x \in A(S) \mid X_S + x = X_S\}$$

where X_S stands for $X \times_k S$ and $+x$ is the translation by x on $A_S = A \times_k S$. In particular, if we put $S = \text{Spec}(k(\varepsilon))$ we have

$$\text{Stab}_A(X)(k(\varepsilon)) = \{x \in A(k(\varepsilon)) \mid X_{k(\varepsilon)} + x = X_{k(\varepsilon)}\}.$$

We write $\text{Stab}_A(X)(k(\varepsilon))_0$ for the set of points in $\{x \in A(k(\varepsilon)) \mid X_{k(\varepsilon)} + x = X_{k(\varepsilon)}\}$ which reduce to the zero of A . This is a subset of the tangent space to A in zero

$$\text{Tang}_0(A) = \{x \in A(k(\varepsilon)) \mid x \text{ reduces to zero}\}.$$

Since $\Omega_{A/k}^\vee \simeq \mathcal{O}_A^{\dim A}$, we have an isomorphism $\text{Tang}_0(A) \simeq H^0(A, \Omega_{A/k}^\vee)$ and hence

$$\text{Stab}_A(X)(k(\varepsilon))_0 \subseteq H^0(A, \Omega_{A/k}^\vee).$$

Identifying $H^0(A, \Omega_{A/k}^\vee)$ with the set of deformations of the identity on A , the inclusion above is given by

$$x \mapsto +x : A_{k(\varepsilon)} \rightarrow A_{k(\varepsilon)}.$$

It is clear then that $\text{Stab}_A(X)(k(\varepsilon))_0$ corresponds exactly to the set of deformations of the identity on A coming from deformations of the identity on X . By our hypothesis $\text{Stab}_A(X)$ is trivial, therefore $\text{Stab}_A(X)(k(\varepsilon))_0 = 0$. Equivalently

$$H^0(X, \Omega_{X/k}^\vee) = 0.$$

□

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